Celestial Mechanics

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Outline

Introduction
What do we observe?

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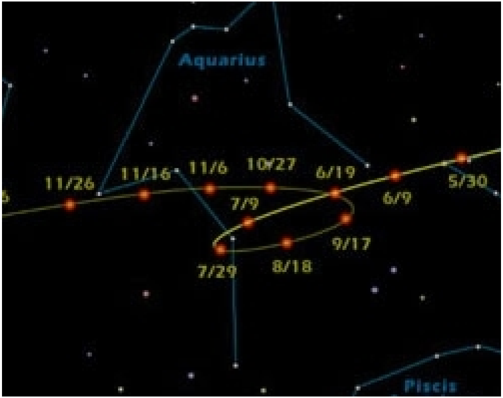
- Ancient man might have started watching the sky out of pure boredom. And so in all its glory, began **astronomy**...
- He would have started classifying objects into those that move with respect to the sky background and those that don't.



Motion of Mars in the sky.

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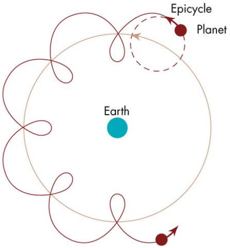

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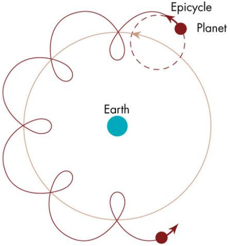

- The complexity of this model increased as we made more accurate observations.

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- The orbit of a planet is an ellipse with the Sun at one of the two foci.
- A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.
- The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.

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- $$m_1 \frac{\mathrm{d}^2 \vec{r_1}}{\mathrm{d}t^2} = \frac{-Gm_1 m_2(\vec{r_1} - \vec{r_2})}{r^3}$$
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- These are in all 6 coupled second order non-linear differential equations.

- We separate the motion into independent components:

$$M\frac{\mathrm{d}^2\vec{R}}{\mathrm{d}t^2} = 0$$
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- We go to plane polar co-ordinates and obtain the following equations:

$$\frac{\mathrm{d}(\mu r^2\dot{\theta})}{\mathrm{d}t} = 0$$
$$\mu\ddot{r} - \mu\dot{\theta}^2 r = \frac{-Gm_1 m_2}{r^2}$$

- We define the following variables for convenience: $l = \mu r^2 \dot{\theta}$ and $k = Gm_1 m_2$. We get,

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- It is easy to see that this is the equation of a conic section. We have solved the two body problem. Cheers!

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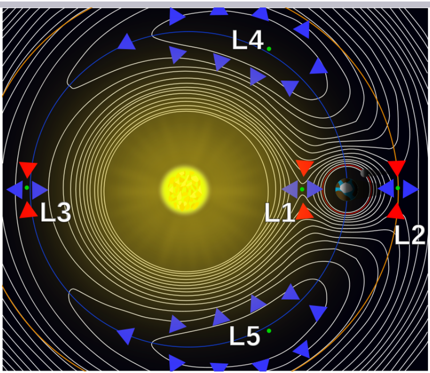
- **Is all hope lost then?**



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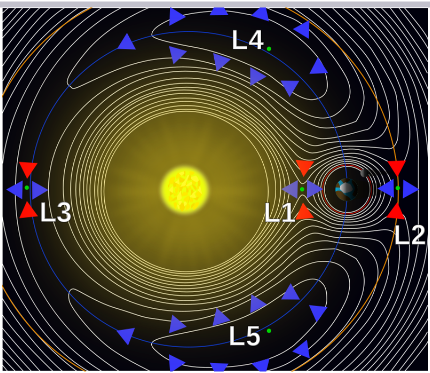

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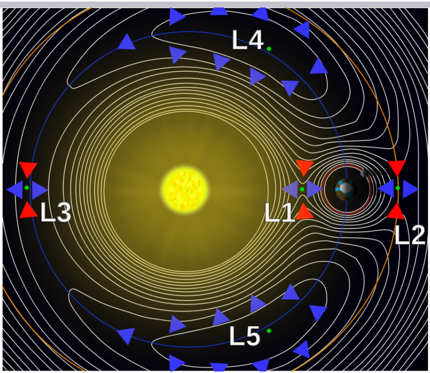

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- A certain type of asteroids known as Trojans are present at *L*4 and *L*5 of Jupiter.

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- Where does Chaos fit in with celestial mechanics?

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This kind of dichotomy wherein the single case looks bad but the system as a whole behaves nicely is echoed across physics.

- For example, compare quantum mechanical description of a single electron and a solid object as a whole.

That's all folks!

References

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